Reducing CCN to CN and N

Consider CCU(a,b,c), where U ≡ N, in terms of the following five operations:

operation 1	CV(b,c)
operation 2	CN(a,b)
operation 3	$CV^{\dagger}$ (b,c)
operation 4	CN(a,b)
operation 5	CV(a,c)

See the graph on page 3461, Barenco et al. 1995 (also on page 182 of Nielsen and Chuang 2000).

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$V = \tfrac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$ state c: a rotation of 45° into the imaginary plane

$$V^{\dagger} = \tfrac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$ state c: the opposite rotation of 45° into the imaginary plane

$V^2 = U$ [V is the square root of u]

$VV^{\dagger} = I$ [first rotation is reversed by the second rotation]

case 1: 0, 0

no operation at all (none of the five apply)

I

case 2: 0, 1

only operations 1 and 3 apply

$VV^{\dagger} = I$

case 3: 1, 0

operations 2, 3, 4, and 5 apply

1 will not apply
applying 2 means that 3 will apply
applying 4 returns the original state a
5 will apply

$V^{\dagger}V = I$

case 4: 1, 1

operations 1, 2, 4, and 5 apply

1 applies
applying 2 turns b off, so 3 will not apply
applying 4 returns the a to 1
now 5 will apply

$VV = U$

So we get U (that is, negation or N) in only case 4, thus CCN. Otherwise, we have only I (that is, no change at all).